

Let \mathcal{P} denote the probability of selecting (r_1, \dots, r_n) as samples, or equivalently, (r_1, \dots, r_n) being rows with the top n largest keys, where r_1 has the largest key, r_2 has the second largest key, etc. We wish to show $\mathcal{P} = \prod_{j=1}^n \left(w_j / \sum_{k=j}^N w_k \right)$.

Recall the key for the j -th row (denoted as x_j from now on) is sampled from a probability distribution on $(-\infty, 0)$ with CDF $F_j(x) = e^{w_j \cdot x}$, and therefore the PDF of x_j is $f_j(x) = F_j'(x) = w_j e^{w_j \cdot x}$. Given that $x_1 \geq x_2 \geq \dots \geq x_n$, and also, $x_n \geq x_j$ for $j \in \{n+1, \dots, N\}$, we have

$$\mathcal{P} = \int_{-\infty}^0 f_1(x_1) \int_{-\infty}^{x_1} f_2(x_2) \cdots \int_{-\infty}^{x_{n-1}} f_n(x_n) \int_{-\infty}^{x_n} f_{n+1}(x_{n+1}) \cdots \int_{-\infty}^{x_n} f_N(x_N) dx_N \cdots dx_2 dx_1$$

Working through the multiple integrals above with ‘ \dots ’ in between and with intermediate steps containing more ‘ \dots ’s would be a bit too hand-wavy! So, in the interest of greater clarity, let’s be more pedantic and re-define \mathcal{P} iteratively with $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_n$ instead. Let

$$\begin{aligned} \mathcal{P}_0(x_n) &= \int_{-\infty}^{x_n} f_{n+1}(x_{n+1}) \cdots \int_{-\infty}^{x_n} f_N(x_N) dx_N \cdots dx_{n+1} \\ &= \prod_{j=n+1}^N \left(\int_{-\infty}^{x_n} f_j(x_j) dx_j \right) = \prod_{j=n+1}^N \left(F_j(x) \Big|_{-\infty}^{x_n} \right) \\ &= \prod_{j=n+1}^N \left(e^{w_j \cdot x} \Big|_{-\infty}^{x_n} \right) = e^{\left(\sum_{j=n+1}^N w_j \right) \cdot x_n} \end{aligned}$$

(i.e., $\mathcal{P}_0(x_n)$ is the inner-most bunch of integrals where the integrands are functions of x_{n+1}, \dots, x_N),

and then define

$$\mathcal{P}_j(x_{n-j}) = \int_{-\infty}^{x_{n-j}} f_{n-j+1}(x_{n-j+1}) \mathcal{P}_{j-1}(x_{n-j+1}) dx_{n-j+1}$$

for $j \in \{1, \dots, n-1\}$, then it follows

$$\mathcal{P}_n(x_0) = \int_{-\infty}^{x_0} f_1(x_1) \mathcal{P}_{n-1}(x_1) dx_1$$

and $\mathcal{P} = \mathcal{P}_n(0)$.

Let's then find out what \mathcal{P}_1 is. By definition:

$$\begin{aligned}
\mathcal{P}_1(x_{n-1}) &= \int_{-\infty}^{x_{n-1}} f_n(x_n) \mathcal{P}_0(x_n) dx_n = \int_{-\infty}^{x_{n-1}} w_n \cdot e^{w_n \cdot x_n} \left[e^{\left(\sum_{j=n+1}^N w_j \right) \cdot x_n} \right] dx_n \\
&= w_n \int_{-\infty}^{x_{n-1}} e^{\left(\sum_{j=n}^N w_j \right) \cdot x_n} dx_n = \left[w_n / \left(\sum_{j=n}^N w_j \right) \right] \cdot e^{\left(\sum_{j=n}^N w_j \right) \cdot x_n} \Big|_{-\infty}^{x_{n-1}} \\
&= \left[w_n / \left(\sum_{j=n}^N w_j \right) \right] \cdot e^{\left(\sum_{j=n}^N w_j \right) \cdot x_{n-1}}
\end{aligned}$$

Now all that is remaining is simply an exercise of proof by mathematical induction, where given the induction hypothesis

$$\mathcal{P}_j(x_{n-j}) = \left[\prod_{h=n-j+1}^n \left(w_h / \sum_{k=h}^N w_k \right) \right] \cdot e^{\left(\sum_{k=n-j+1}^N w_k \right) \cdot x_{n-j}}$$

, which is true for $j = 1$, we shall show it is true for $j \in \{2, \dots, n\}$.

Suppose the induction hypothesis is true for $j = \mathcal{I} - 1$, then by definition

$$\begin{aligned}
\mathcal{P}_{\mathcal{I}}(x_{n-\mathcal{I}}) &= \int_{-\infty}^{x_{n-\mathcal{I}}} f_{n-\mathcal{I}+1}(x_{n-\mathcal{I}+1}) \mathcal{P}_{\mathcal{I}-1}(x_{n-\mathcal{I}+1}) dx_{n-\mathcal{I}+1} \\
&= \int_{-\infty}^{x_{n-\mathcal{I}}} w_{n-\mathcal{I}+1} \cdot e^{w_{n-\mathcal{I}+1} \cdot x_{n-\mathcal{I}+1}} \cdot \left[\prod_{h=n-\mathcal{I}+2}^n \left(w_h / \sum_{k=h}^N w_k \right) \right] \cdot e^{\left(\sum_{k=n-\mathcal{I}+2}^N w_k \right) \cdot x_{n-\mathcal{I}+1}} dx_{n-\mathcal{I}+1} \\
&= w_{n-\mathcal{I}+1} \left[\prod_{h=n-\mathcal{I}+2}^n \left(w_h / \sum_{k=h}^N w_k \right) \right] \cdot \int_{-\infty}^{x_{n-\mathcal{I}}} e^{\left(\sum_{k=n-\mathcal{I}+1}^N w_k \right) \cdot x_{n-\mathcal{I}+1}} dx_{n-\mathcal{I}+1} \\
&= \left[w_{n-\mathcal{I}+1} / \left(\sum_{k=n-\mathcal{I}+1}^N w_k \right) \right] \left[\prod_{h=n-\mathcal{I}+2}^n \left(w_h / \sum_{k=h}^N w_k \right) \right] \cdot \left[e^{\left(\sum_{k=n-\mathcal{I}+1}^N w_k \right) \cdot x_{n-\mathcal{I}+1}} \Big|_{-\infty}^{x_{n-\mathcal{I}+1}} \right] \\
&= \prod_{h=n-\mathcal{I}+1}^n \left(w_h / \sum_{k=h}^N w_k \right) \cdot e^{\left(\sum_{k=n-\mathcal{I}+1}^N w_k \right) \cdot x_{n-\mathcal{I}}}
\end{aligned}$$

which shows the induction hypothesis is true for $j = \mathcal{I}$.

Therefore

$$\mathcal{P}_j(x_{n-j}) = \left[\prod_{h=n-j+1}^n \left(w_h / \sum_{k=h}^N w_k \right) \right] \cdot e^{\left(\sum_{k=n-j+1}^N w_k \right) \cdot x_{n-j}}$$

for $j \in \{1, \dots, n\}$ and

$$\mathcal{P} = \mathcal{P}_n(0) = \left[\prod_{h=1}^n \left(w_h / \sum_{k=h}^N w_k \right) \right] \cdot e^{\left(\sum_{k=1}^N w_k \right) \cdot 0} = \prod_{h=1}^n \left(w_h / \sum_{k=h}^N w_k \right)$$

QED.